CHAPTER ONE: INTRODUCTION

1.1 Origin of Operations Research

The term Operations Research (OR) was first coined by MC Closky and Trefthen in 1940 in a small town, Bowdsey of UK. The main origin of OR was during the Second World War – The military commands of UK and USA engaged several inter-disciplinary teams of scientists to undertake scientific research into strategic and tactical military operations.

Their mission was to formulate specific proposals and to arrive at the decision on optimal utilization of scarce military resources and also to implement the decisions effectively. In simple words, it was to uncover the methods that can yield greatest results with little efforts. Thus it had gained popularity and was called "An art of winning the war without actually fighting it"

The name Operations Research (OR) was invented because the team was dealing with research on military operations. The encouraging results obtained by British OR teams motivated US military management to start with similar activities. The work of OR team was given various names in US: Operational Analysis, Operations Evaluation, Operations Research, System Analysis, System Research, Systems Evaluation and so on.

The first method in this direction was simplex method of linear programming developed in 1947by G.B Dantzig, USA. Since then, new techniques and applications have been developed to yield high profit from least costs.

Now OR activities has become universally applicable to any area such as transportation, hospital management, agriculture, libraries, city planning, financial institutions, construction management and so forth. In India many of the industries like Delhi cloth mills, Indian Airlines, Indian Railway, etc are making use of OR activity.

1.2 Concept and Definition of OR

Operations research signifies research on operations. It is the organized application of modern science, mathematics and computer techniques to complex military, government, business or industrial problems arising in the direction and management of large systems of men, material, money and machines. The purpose is to provide the management with explicit quantitative understanding and assessment of complex situations to have sound basics for arriving at best decisions. Operations research seeks the optimum state in all conditions and thus provides optimum solution to organizational problems.

Definition: OR is a scientific methodology – analytical, experimental and quantitative – which by assessing the overall implications of various alternative courses of action in a management system provides an improved basis for management decisions.

1.3 Characteristics of OR (Features)

The essential characteristics of OR are

1. **Inter-disciplinary team approach** – The optimum solution is found by a team of scientists selected from various disciplines.

2. **Holistic approach to the system** – OR takes into account all significant factors and finds the best optimum solution to the total organization.

3. **Imperfectness of solutions** – Improves the quality of solution.

4. **Use of scientific research** – Uses scientific research to reach optimum solution.

5. **To optimize the total output** – It tries to optimize by maximizing the profit and minimizing the loss.

1.4 Applications of OR

Some areas of applications are

- ❖ Finance, Budgeting and Investment
- ✓ Cash flow analysis, investment portfolios
- ✓ Credit policies, account procedures
- ❖ Purchasing, Procurement and Exploration
- ✓ Rules for buying, supplies
- ✓ Quantities and timing of purchase
- ✓ Replacement policies
- ❖ Production management
- ✓ Physical distribution
- ✓ Facilities planning
- ✓ Manufacturing

- ✓ Maintenance and project scheduling
- ❖ Marketing
- ✓ Product selection, timing
- ✓ Number of salesman, advertising
- ❖ Personnel management
- ✓ Selection of suitable personnel on minimum salary
- ✓ Mixes of age and skills
- ❖ Research and development
- ✓ Project selection
- ✓ Determination of area of research and development
- ✓ Reliability and alternative design

1.5 Phases of OR

OR study generally involves the following major phases

1. Defining the problem and gathering data
2. Formulating a mathematical model
3. Deriving solutions from the model
4. Testing the model and its solutions
5. Preparing to apply the model
6. Implementation

Defining the problem and gathering data

- ✓ The first task is to study the relevant system and develop a well-defined statement of the problem. This includes determining appropriate objectives, constraints, interrelationships and alternative course of action.
- ✓ The OR team normally works in an **advisory capacity**. The team performs a detailed technical analysis of the problem and then presents recommendations to the management.
- ✓ Ascertaining the appropriate **objectives** is very important aspect of problem definition.
- ✓ Some of the objectives include maintaining stable price, profits, increasing the share in market, improving work morale etc.
- ✓ OR team typically spends huge amount of time in gathering relevant data.
- To gain accurate understanding of problem

- To provide input for next phase.
- ✓ OR teams uses Data mining methods to search large databases for interesting patterns that may lead to useful decisions.

Formulating a mathematical model

This phase is to reformulate the problem in terms of mathematical symbols and expressions. The mathematical model of a business problem is described as the system of equations and related mathematical expressions. Thus

1. **Decision variables** (x1, x2 … xn) – ‘n’ related quantifiable decisions to be made.

2. **Objective function** – measure of performance (profit) expressed as mathematical function of decision variables. For example P=3x1 +5x2 + … + 4xn

3. **Constraints** – any restriction on values that can be assigned to decision variables in terms of inequalities or equations. For example x1 +2x2 ≥ 20

4. **Parameters** – the constant in the constraints (right hand side values)

The advantages of using mathematical models are

- ✓ Describe the problem more concisely
- ✓ Makes overall structure of problem comprehensible
- ✓ Helps to reveal important cause-and-effect relationships
- ✓ Indicates clearly what additional data are relevant for analysis
- ✓ Forms a bridge to use mathematical technique in computers to analyze

Deriving solutions from the model

This phase is to develop a procedure for deriving solutions to the problem. A common theme is to search for an optimal or best solution. The main goal of OR team is to obtain an optimal solution which minimizes the cost and time and maximizes the profit.

Herbert Simon says that “Satisficing is more prevalent than optimizing in actual practice”.

Where satisficing = satisfactory + optimizing

Samuel Eilon says that “Optimizing is the science of the ultimate; Satisficing is the art of the feasible”.

To obtain the solution, the OR team uses

- ✓ **Heuristic procedure** (designed procedure that does not guarantee an optimal solution) is used to find a good suboptimal solution.

- ✓ **Metaheuristics** provides both general structure and strategy guidelines for designing a specific heuristic procedure to fit a particular kind of problem.
- ✓ **Post-Optimality analysis** is the analysis done after finding an optimal solution. It is also referred as **what-if analysis**. It involves conducting **sensitivity analysis** to determine which parameters of the model are most critical in determining the solution.

Testing the model

After deriving the solution, it is tested as a whole for errors if any. The process of testing and improving a model to increase its validity is commonly referred as **Model validation**. The OR group doing this review should preferably include at least one individual who did not participate in the formulation of model to reveal mistakes.

A systematic approach to test the model is to use **Retrospective test**. This test uses historical data to reconstruct the past and then determine the model and the resulting solution. Comparing the effectiveness of this hypothetical performance with what actually happened indicates whether the model tends to yield a significant improvement over current practice.

Preparing to apply the model

After the completion of testing phase, the next step is to install a well-documented system for applying the model. This system will include the model, solution procedure and operating procedures for implementation.

The system usually is computer-based. **Databases** and **Management Information System** may provide up-to-date input for the model. An interactive computer based system called **Decision Support System** is installed to help the manager to use data and models to support their decision making as needed. A **managerial report** interprets output of the model and its implications for applications.

Implementation

The last phase of an OR study is to implement the system as prescribed by the management. The success of this phase depends on the support of both top management and operating management.

The implementation phase involves several steps

1. OR team provides a detailed explanation to the operating management

2. If the solution is satisfied, then operating management will provide the explanation to the personnel, the new course of action.

3. The OR team monitors the functioning of the new system
4. Feedback is obtained
5. Documentation

CHAPTER 2: LINEAR PROGRAMING (LP)

2.1. Basic Concepts in LP

Linear programming deals with the optimization (maximization or minimization) of a function of variables known as objective function, subject to a set of linear equation and /or inequalities known as constraints.

- The objective function may be profit, cost, production capacity, or any other measure of effectiveness which is to be obtained in the best possible or optimal manner
- The constraints may be imposed by different resources such as market demand, production process and equipment, storage capacity, raw material availability, etc.
- By linearity is meant a mathematical expression in which the expressions among the variables are linear.

Definition:

LP is a mathematical modelling technique useful for economic allocation of “scarce” or “limited” resources (such as labour, material, machine, time, warehouse, space, capital, etc.) to several competing activities (such as products, services, jobs, new equipment, projects etc.) on the basis of a given criterion of optimality.
All organizations, big or small, have at their disposal, men, machines, money and materials, the supply of which may be limited. Supply of resources being limited, the management must find the best allocation of resources in order to maximize the profit or minimize the loss or utilize the production capacity to the maximum extent.

Steps in formulation of LP model:

1. Identify activities (key decision variables).
2. Identify the objective function as a linear function of its decision variables.
3. State all resource limitations as linear equation or inequalities of its decision variables.
4. Add non-negative constraints from the consideration that negative values of the decision variables do not have any valid physical interpretation.
5. Use mathematical techniques to find all possible sets of values of the variables (unknowns) satisfying all the function
6. Select the particular set of values of variables obtained in step five that leads to the achievement of the objective function.

- The result of the first four steps is called linear programming. The set of solutions obtained in step five is known as the set of feasible solutions and the solution finally selected in step six is called optimum (best) solution of the LP problem.

- A typical linear programming has two step
  - ✓ The objective function
  - ✓ The constraints

- Technical constraint, and
- The non-negativity constraint

The objective function: is a mathematical representation of the overall goal of the organization stated as a linear function of its decision variables (X_j) to optimize the criterion of optimality.

- It is also called the measure of performance such as profit, cost, revenue, etc

The general form of the objective function is expressed as:
Optimize (Maximize or Minimize) Z=$\sum C_j X_j$
Where: Z is the measure of performance variable (profit/cost), which is the function of X_1, X_2, …,X_n. (Quantities of activities)
Cj (C_1, C_2… C_n) (parameters or coefficients) represent the contribution of a unit of the respective variables X_1, X_2, …,X_n to the measure of performance Z (the objective function).

The constraints: the constraints must be expressed linear equalities or inequalities interms of decision variables.
The general form of the constraints functions are expressed as:

$$\sum a_{ij} x_j \quad (<,>, \leq, =, \geq)\ b_i$$

$a_{11}X_1+a_{12}X_2+\ldots+a_{1n}X_n\ (<,>, \leq, =, \geq)\ b_1$
$a_{21}X_1+a_{22}X_2+\ldots+a_{2n}X_n\ (<,>, \leq, =, \geq)\ b_2$
. . .
. . .
. . .
$A_{m1}X_m+a_{m2}X_2+\ldots+a_{mn}X_n\ (<,>, \leq, =, \geq)\ b_n$

- a_{ij}'s are called technical coefficients and measure the per unit consumption of the resources for executing one unit of unknown variable (activities) X_j.
    - a_{ij}'s can be positive, negative or zero in the given constraints.

- The b_i represents the total availability of the i^{th} resource.
    - It is assumed that $b_i \geq 0$ for all i. However, if any $b_i < 0$, then both sides of the constraint i can be multiplied by 1 to make $b_i > 0$ and reverse the inequality of the constraint.

Requirements for a linear programming problem: Generally speaking, LP can be used for optimization problems if the following conditions are satisfied.

1. There must be a well defined objective function which is to be either maximized or minimized and which can be expressed as a linear function of decision variables
2. There should be constraints on the amount or extent of attainment of the objective and these constraints must be capable of being expressed as linear equations or inequalities interms of variables.

3. There must be alternative courses of action. Fore e.g., a given product may be processed by two different machines and problem may be as to how much of the product to allocate which machine.
4. Decision variables should be interrelated and non-negative.
5. The resource should be in limited supply.

2.2. Assumptions of Linear Programming

A linear programming model is based on the following assumptions:

1. Proportionality assumption: A basic assumption of LP is that proportionality exists in the objective function and the constraints. It states that the contribution of each activity to the value of the objective function Z is proportional to the level of the activity X_j, as represented by the CjX_j term in the objective function. Similarly, the resource consumption of each activity in each functional constraint is proportional to the level of the activity X_j, as represented by the $a_{ij}X_j$ term in the constraint.
   What happens when the proportionality assumption does not hold? In most cases we use nonlinear programming.

2. Additivity assumption: States that every function in a linear programming model (whether the objective function or the left-hand side of the a functional constraint)is the sum of the individual contributions of the respective activities.
   What happens when the additivity assumption does not hold? In most cases we use nonlinear programming.

3. Divisibility assumption: States that decision variables in linear programming model are allowed to have any values, including non-integer values, which satisfy the functional and non-negativity constraints. Thus, since each decision variable represents the level of some activity, it is being assumed that the activities can be run at fractional levels.
   In certain situations, the divisibility assumption does not hold because some of or all the decision variables must be restricted to integer values. For this restrictions integer programming model will be used.

4. Certainity assumption: This assumption states that the various parameters (namely, the objective function coefficients, the coefficients in the functional constraints a_{ij} and resource values in the constraints b_i are certainly and precisely known and that their values do not change with time. However, in real applications, the certainity assumption is seldom satisfied precisely. For this reason it is usually important to conduct sensitivity analysis after a solution is found that is optimal under the assumed parameter values.

5. Finiteness: An LP model assumes that a finite (limited) number of choices (alternatives) are available to the decision-maker and that the decision variables are interrelated and non negative. The non-negativity condition shows that LP deals with real-life situations as it is not possible to produce/use negative quantities.

6. Optimality: In LP, the optimal solution always occurs at the corner point of the set of feasible solutions.

Important Definitions

- Solution: The set of values of decision variables X_i (i=1, 2, ….,n)which satisfy the constraints of an LP problem is said to constitute solution to that LP problem
- Feasible solutions: The set of values of decision variables X_i (i=1, 2… n) which satisfy all the constraints and non-negativity conditions of an LP simultaneously.
- Infeasible solution: The set of values of decision variables which do not satisfy all the constraints and non-negativity conditions of an LP simultaneously.
- Optimum solution: A feasible solution which optimizes (maximizes or minimizes) the objective function value of the given LP problem is called an optimum basic feasible solution.
- Unbounded solution: A solution which can increase or decrease the value of the objective function of LP problem indefinitely is called unbounded solution.

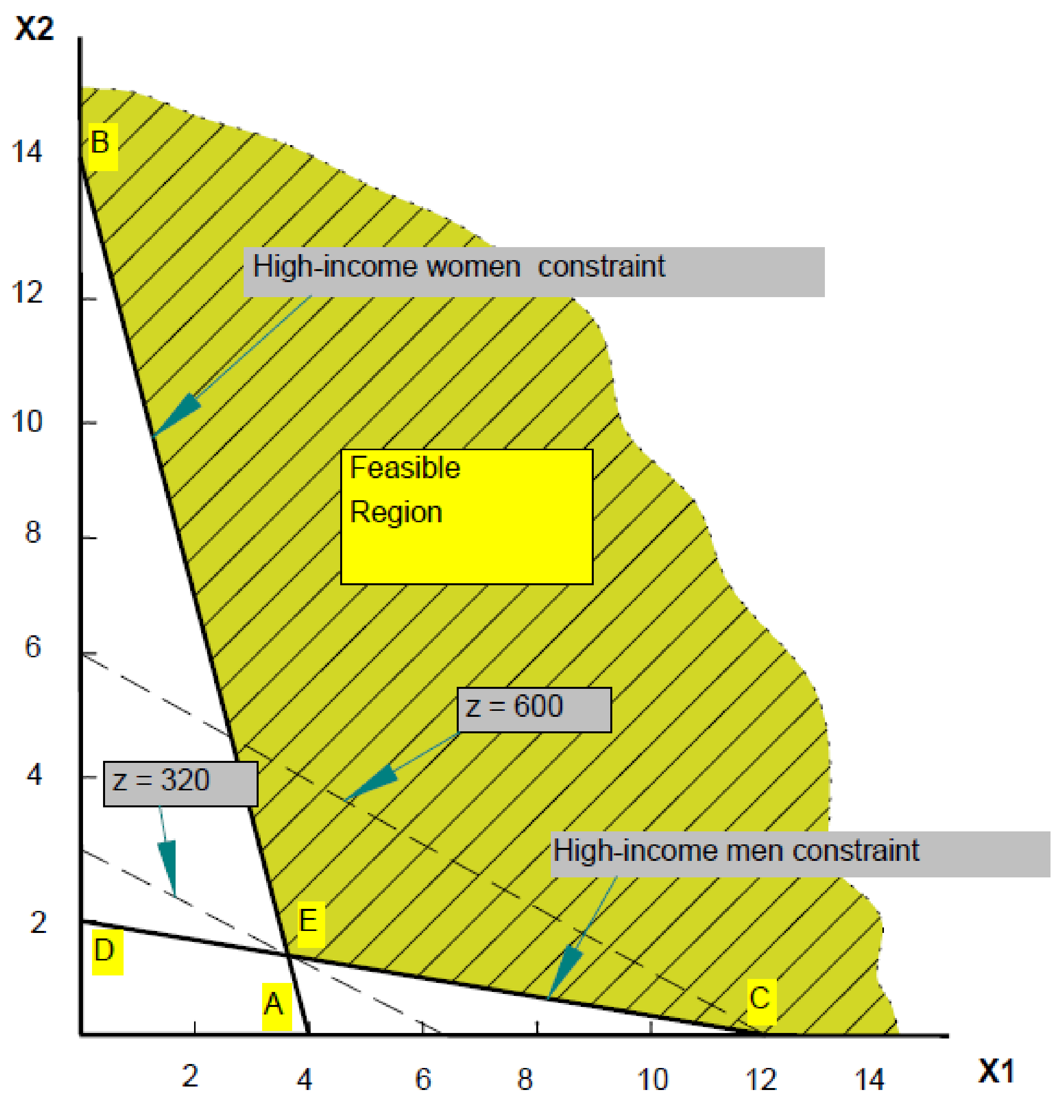
2.3. Methods of Solving Linear Programming Problems

A linear programming problem can be solved by graphic method or by applying algebraic method, called the simplex method.

1. Graphic method: it can be used when only two variables are involved. This method consists of the following steps:

i. Draw coordinate space Represent the given problem in mathematical form draw the coordinate points.
ii. Draw constraint-lines Plot each of the constraint on the graph.
iii. Define feasible region Identify the feasible region (or solution space) that satisfies all the constraints simultaneously. Any point on or within the shaded region represents a feasible solution to the given problem. Locate the solution points (identify the corner points from the solution space).
iv. Determine the optimal solution. A solution that optimizes the objective function.
v. Interpret the results

Example:-min $z = 50x_1 + 100x_2$
s.t. $7x_1 + 2x_2 \geq 28$ (high income women)
$2x_1 + 12x_2 \geq 24$ (high income men)
$x_1, x_2 \geq 0$



Example1. Solve the following LLP using graphic method,

Max $Z = 2x_1 + 3x_2,$

Subject to:

$4x_1 + 3x_2 \leq 18$

$5x_1 + 2x_2 \leq 19$

$x_1, \ x_2 \geq 0$

Example2. A firm manufactures two products A and B on which the profits earned per unit are Birr 3 and Birr 4 respectively. Each product is processed on two machines M_1 and M_2. Product A requires one minute of processing time on M_1 and two minutes on M_2, while B requires one minute on M_1 and one minute on M_2. Machine M_1 is available for not more than 7hrs. 30 minutes, while machine M_2 is available for 10hrs during any working day. Find the number of units of products A and B to be manufactured to get maximum profit.

Solution

Formulation of LPP model

Let x_1 and x_2 denote the number of units of products A and B to be produced per day.

Objective is to maximize the profit.

i.e. Maximize $Z = 3x_1 + 4x_2$

Constraints are on the time available for machines M_1 and M_2

i.e., for machine M_1, $1x_1 + 1x_2 \leq 450$

for machine M_2, $2x_1 + 1x_2 \leq 600$ Ans: x1=0, x2=450 and Z_{max} =Birr 1800

Where $x_1, x_2 \geq 0$.

Example 3: The standard weight of a special purpose brick is 5kg and it contains two basic ingredients B_1 and B_2. B_1 costs Birr 5/kg and B_2 costs Birr 8/kg. Strength considerations dictate that the brick contains not more than 4kg of B_1 and a minimum of 2kg of B2. Since the demand for the product is likely to be related to the price of the brick, find graphically the minimum cost of the brick satisfying the above conditions.

Solution

Formulation of LPP model

Let the quantities in kg of ingredients B1 and B2 to be used to make the bricks be x1 and x2 resp.

Objective is to minimize the cost of the brick.

i.e., Minimize $Z = 5x_1 + 8x2$ Ans: x_1=3kg, x2= 2kg, Z_{min}=Birr 31

Constraints are

On the quantity of the ingredient B_1: $x_1 \leq 4$,

On the quantity of the ingredient B_2: $x_2 \geq 2$,

On the weight of the brick : $x_1 + x_2 = 5$

Where $x_1, x_2 \geq 0$.

Some Special Cases of Linear Programming

There are certain special cases in linear programming problem. Some of these cases are resulted from violations of the basic assumptions of LP. These include:

1. Multiple optimal solutions: This is the situation when more than one optimal solution arises. This will happen whenever the objective function equation represents a line parallel to some edge of the bounded solution space.

Example: Solve the following problems graphically

Max $Z = 110x_1+110x_{2,}$

Subject to:

$x_1 + x_2 \leq 9$,

$x_1 \geq 2$,

$20x_1 + 50x_2 \leq 360$,

$x_2 \geq 3$

$x_1, x_2 \geq 0.$

2. Unbounded solution: it exists when an LP problem has no limit on the constraints i.e., the feasible region is not bounded in any respect. Theoretically, the value of the decision variable increases infinitely with out violating the feasibility and value of the objective function can increase to infinity. Mostly this situation occurs when the assumption of finiteness is violating.

Example: Solve the following problems graphically

$\text{Max } Z = 30x_1+50x_2,$

Subject to:

$3x_1 + x_2 \geq 9,$

$x_1 + 2x_2 \geq 12,$

$x_1 + 2x_2 \geq 9,$

$x_1, x_2 \geq 0.$

3. Infeasible solution: infeasibility is a condition that exists when there is no solution to an LP problem that satisfies all the constraints and non-negativity restrictions. It means that the constraints in the problem are conflicting and inconsistent. Infeasibility depends solely on the constraints and has nothing to do with the objective function.

Example: Solve the following problems graphically

$\text{Max } Z = 3x_1+2x_2,$

Subject to:

$-2x_1 + 3x_2 \leq 9,$

$3x_1 - 2x_2 \leq -20,$

$x_1, x_2 \geq 0.$

Binding and Non-Binding Constraints (Resources)

Once the optimal solution to an LPP is obtained, we may classify the constraints as being binding and or non-binding. A constraint is termed as binding if the left hand side and the right hand side of the constraint function s are equal when the optimal values of the decision variables are substituted in to the constraint. On the other hand, if the substitutions of the value of the decision variables do not lead to equality between the left and the right hand side of the constraint, it said to be non-binding.

Redundant constraint(s): If the constraint, when plotted does not form part of the boundary marking the feasible region of the problem, it is said to be redundant. The inclusion or the exclusion of a redundant constraint obviously does not affect the optimal solution to the problem.

Exercises: Solve the following LLPs using graphic method

1. a) $\text{Min } Z = 2x_1+x_2,$

b) $\text{Max } Z = 2x_1+3x_2$ — Ans. a) $x_1=0, x_2=2, Z_{min}=2$, b) Unbounded

Subject to:

$x_1 - 3x_2 \leq 6,$

$2x_1 + 4x_2 \geq 8,$
$x_1 - 3x_2 \geq -6,$
$x_1, x_2 \geq 0.$

2. Max $Z = 4x_1+5x_2,$ Ans. Unbounded Solution

Subject to:
$x_1 + x_2 \geq 1,$
$-2x_1 + x_2 \leq 1,$
$4x_1 - x_2 \geq 1$
$x_1, x_2 \geq 0.$

3. A company produces two products P1 and P2. The products are produced and sold on a weekly basis. Because of limited available resources, the weekly production can't exceed 25 units for product P1 and 35 units for product P2. The company employs a total of 60 workers. Product P1 requires 2 man-weeks of labour, while P2 requires one man-week of labour. Profit margin on P1 is 60 Birr and on P2 is Birr 40.

   a. Formulate the above problem as linear programming problem
   b. Find the product mix of the two products so as to maximize the profit of the company

4. Giapetto's wooden soldiers and trains. Each soldier sells for \$27, uses \$10 of raw materials and takes \$14 of labour & overhead costs. Each train sells for \$21, uses \$9 of raw materials, and takes \$10 of overhead costs. Each soldier needs 2 hours finishing and 1 hour carpentry; each train needs 1 hour finishing and 1 hour carpentry. Raw materials are unlimited, but only 100 hours of finishing and 80 hours of carpentry are available each week. Demand for trains is unlimited; but at most 40 soldiers can be sold each week. How many of each toy should be made each week to maximize profits?

2. The Simplex Method

When a number of variables in a linear programming are more than two, graphic method cannot be used because of the difficulty precisely representing the variables using more than a two dimensional plane. That means it is impossible to represent these variables in the graph and evaluate the corner points. In such cases, there is a comprehensive method of solving a linear programming problem which is called the simplex method (simplex algorithm).

- Algorithm is a process where a systematic procedure is repeated (iterated) over and over again until the desired result is obtained.

Simplex method (algorithm) is an iterative procedure that consists of moving from one basic feasible to another in such a way that the values of the objective function doesn't decrease (in case of maximization problem) and doesn't increase (in case of minimization problem).

- The process continues until an optimal solution is reached if it exists, otherwise, the problem may be unbounded or not feasible and the simplex method can't find the solution.

Conditions for applications of simplex method

There are some conditions that must be fulfilled for the application of simplex method. These are:

1. The right hand side of each of the constraints bi should be non-negative.

   - If the linear programming problem has a negative constraint, we should convert it to positive by multiplying both sides by -1. Fore example, if one of the constraints is given by $2x_1 - 4x_2 \geq -8$, we cannot apply simplex method as it is. So, we should convert it to positive by multiply both sides by -1. i.e., $-2x_1 + 4x_2 \leq 8$

2. Each of the decision variables of the problem should be non-negative. If one of the choice variables is not feasible, simplex method cannot be applied. Therefore, feasibility is a necessary condition for application of simplex method.

3. The inequality constraints of resources or any other activities must be converted in to equations.

Since simplex method cannot be applied in the case of inequalities, it is a mandatory that these inequalities should be converted in to equations. There are some variables that help us to convert these inequalities in to equations. These variables are: Slack and surplus variables.
Slack variables: are variables that can change the less than type inequality in to equations.

- Slack variables represent unused amount of a given resource by the activities or idle resources. For example, if one of the constraint is given as:
  - $x_1 \leq 4$ we can convert this in to equations by introducing a slack variable.
  - $x_1 + S_1 = 4$. The values of S_1 will range from zero to four. i.e., if $x_1=0$, $S_1=4$ meaning that all the available resources is not used. If $x_1=4$, $S_1=0$ meaning that all the available resource is fully utilized and there is no idle resource which is left over.

Surplus variables: if the constraint inequality is of greater than type, then we can convert it into equation by subtracting some variable called surplus variable. Surplus variables represent that the requirement of the resource is more than the availability.

- Surplus variables are the negative of the slack variables. For example, if one of the constraint is given as:
  $x_2 \geq 8$, this inequality can be converted to equation by introducing surplus variable as
  $x_2 - S_1 = 8$.

Note: Slack and surplus variables enter in to the objective function with zero coefficients.

Procedures of simplex method

Step 1: Express the problem in standard form.

The given problem is said to be expressed in standard form if the decision variables are non-negative, R.H.S of the constraints are non-negative and the constraints are expressed as equations. Non-negative slack variables are added to the L.H.S of the constraints to convert them in to equations.

Example: Max $Z = 3x_1+4x_{2,}$

Subject to:

$x_1 + x_2 \leq 450,$

$2x_1 +x_2 \leq 600,$

Where $x_1, x_2 \geq 0.$

Standard form of the above LP model is:

Max $Z = 3x_1+4x_2 +0S_1 +0S_{2,}$

Subject to:

$x_1 + x_2 + S1=450,$

$2x_1 + x_2 +S2= 600,$

Where $x_1, x_{2,} S_1, S_2 \geq 0$

Step 2: Set up the initial solution

- In order to set up the initial solution, first we need to determine the basic and non-basic variables.
- Slack variables at initial table become basic variables and decision variables are non-basic variables. If we have n unknowns and m constraints, then m variables should be basic and the rest (n-m) variables should be non-basic such that their value is zero.
- Basic variables are those variables whose values are obtained from the basic solution where as non-basic variables are those whose values are assumed as zero in the basic solution.

Example: The initial solution of the above standard form can be expressed in the following simplex table

Contribution/unit Cj		3	4	0	0	
		Body matrix		Identity matrix		
C_B	BV	x_1	x_2	s_1	s_2	b
0	s_1	1	1	1	0	450
0	s_2	2	1	0	1	600

Interpretation:

1. The first row indicates the coefficients Cj of the variables in the objective function and they are unchanged in the subsequent tables.
2. The first column C_B represents the coefficients of the current basic variables in the obj. fun. The second column is the basis column and it represents the basic variables of the current solution.

3. The body matrix (coefficient matrix) represents the coefficients of the constraints. These coefficients represent the amount of resource required to make a unit of decision variable.
4. The identity matrix represents the coefficients of slack variables in the constraints.
5. The b-column/solution values x_B column indicates the quantities of the variable resources or R.H.S values of the constraints or values of the basic variables, S_1 and S_2 in the initial basic solution.

Step 3: Perform optimality test

The test for optimality can be based on interms of the sign of the members of the index row.

Contribution/unit Cj		3	4	0	0	
		Body matrix		Identity matrix		
C_B	BV	x_1	x_2	s_1	s_2	B
0	S1	1	(1)	1	0	450⇒key row
0	S2	2	1	0	1	600
	Z_j	0	0	0	0	0
	$\Delta j = Z_j - Cj$	-3	-4	0	0	

↑K

Where: $Z_j = \sum C_{Bi} a_{ij}$ and $Z = \sum C_{Bi} x_{Bi}$

- Values in the Z_j represent the contribution lost per unit of the variables. i.e., the amounts by which contribution would be reduced if one of the corresponding variables x_1, x_2 … was added.

- Δ_j is also called the index row. This row determines whether or not the current solution is optimal. Coefficients in this row represent the net profit (or net contribution or net marginal improvement) in the values of the objective function Z for each unit of the respective column variable introduced into the solution. For example, as x_1 increases by one unit Z will increase by 3 units.

- A positive coefficient in the Z_j-C_j row indicates the amount by which the objective function will be decreased if a unit of the corresponding variable is introduced in to the solution where as a negative coefficient indicates the amount by which the profit will be increased if a unit of the corresponding variable is introduced in to the solution. These coefficients or elements are known as shadow prices or accounting values or imputed values of the resources.

- The solution is optimal if all the elements of the index row are +ve and zero. If at least one of the elements of the index row is -ve, the given solution is not optimal and goes to the next step. The next iteration will give as the next adjusted basic feasible solution.

Step 4: Iterate towards the optimal solution

In each iteration, the simplex method moves the current basic feasible solution to an improved basic feasible solution. This is done by replacing one current basic variable by a new non-basic variable as follows.

- Pivot column: The column which has the maximum -ve value of Z_j-$C_{j,}$ is the one which should enter the solution (entering variable).

- Pivot row: The element lying at the intersection of key column and key row is called key /pivot element and is enclosed in ( ). This pivot element corresponds to the leaving variables. Leaving variable= (xb_i /a_{ij}, a_{ij}) <0.

To find the new solution:

- If the key element (pivot number) is 1, then the row remains the same in the new simplex table

- If the key element (pivot number) is other than 1, then divide each element in the pivot row (including elements in x_b column) by the pivot number, to find the new values of the row in the new simplex table.

- The new values of the elements in the remaining rows for the new simplex table can be obtained by performing elementary row operations on all rows so that all elements except the key element in the key column are zero. i.e.,

  No. in new row = No. in old row - (No. above or below key element) (corresponding No. in old row)

- If all ratios are negative or infinity, the solution is unbounded.
- The next solution will be feasible only if row containing minimum (non-negative) ratio is selected as key row; in case a row containing higher (non-negative) ratio is marked as key row, the next solution will become infeasible and there will be some negative value in the b-column which is illogical.

Contribution/unit		3	4	0	0	
C_j		Body matrix		Identity matrix		
C_B	BV	x_1	x_2	s_1	s_2	B
4	X_2	1	1	1	0	450
0	S_2	1	o	-1	1	150
	Z_j	4	4	4	0	1800 $\Rightarrow 2^{nd}$ f. solution
	$\Delta j = Z_j - C_j$	1	0	4	0	

↑Marginal loss value ↑ Shadow prices

Step 5: Perform optimality test for second feasible solution

If there is a negative number in the index row, then the given solution is not optimal. So we proceed to the step 4 again. However, values in the index row are non-negative the given solution is optimal. Hence the optimal solution is $x_1=0$, $x_2= 450$, $Z_{max}=1800$.

Remarks:

1. The optimal solution obtained above satisfies the non-negativity restrictions. It indicates that no other set of values of x_1 and x_2 results in as high value of Z.

2. Slack variable s_1 does not exist in the basis column. This means that the first resource is fully utilized. However, s_2 exists in the basis column and its value is 150. This means 150 units of the second resources remains unutilized.

3. Elements in the Z_j- Cj row under slack variables indicate the shadow price (or accounting values or imputed values) of resources.
   - ✓ 4 and 0 are called the shadow prices of the first and the second resources respectively.
   - ✓ 4 under s_1 column indicate that if one more unit of the first resource is used, then the value of Z will increase by 4 units. Therefore, if we want to increase Z, it should increase the first resource.
   - ✓ Capacity of the second resource has not been utilized fully; there will be no use increasing it further.

Exercises: Solve the following problems by simplex method and interpret the results.

1. Max $Z = 40x_1+80x_2$

   Subject to: Ans. $x_1=9$ units, $x_2=10$ units, $Z_{max}=1160$

   $2x_1 + 3x_2 \leq 48,$

   $x_1 \leq 15,$ $s_1=20$, $s_2=0$, $s_3=20$

   $x_1 \leq 10,$

   $x_1, x_2 \geq 0.$
2. Max $Z = 2x_1+5x_2$

   Subject to: Ans. $x_1=0$units, $x_2=380/9$ units, $x_3=470/3$ units, $Z_{max}=3200/3$

   $x_1 + 4x_2 \leq 24,$ $s_1=0$, $s_2=0$, $s_3=1970/9$

   $3x_1 + x_2 \leq 21,$

   $x_1 + x_2 \leq 9,$

   $x_1, x_2 \geq 0.$

2.4.Artificial Variables Technique

There are many linear programming problems where slack variables cannot provide a solution. In these problems at least one of the constraints is of (≥) or (=) type. In such cases we introduce another type of variables called artificial variables.

- ➢ Artificial variables are fictitious and have no any physical or economic meaning. They assume the role of surplus variables in the first iteration, only to be replaced at a later iteration.

- Artificial variables are merely a device to get the starting basic feasible solution and then the usual simplex procedure will be adopted until the optimal solution is obtained.

- Artificial variables are expressed by capital letter "A" and used for "≥" type of inequalities and "=" equations.

- Artificial variables enter in to the objective function with "M" coefficient i.e., it inter in to the objective function with –ve value (-M) for maximization and +ve value (+M) for minimization, where M >0.

- M denotes some huge positive number (very huge penalty), for that firm manipulates the artificial variables.

Steps of the big M-method algorithm

Step 1: Express the LP problem in the standard form by introducing slack variables.

  - These variables are added to the L.H S of the constraints of (≤) and subtracted from the constraints of (≥) type.

Step 2: Add non-negative artificial variable (A) corresponding to constraints having "≥" and "=" equations.

Step 3: Set up the initial solution

  - In order to set up the initial solution, first we need to determine the basic and non-basic variables. In the initial solution, original variables have zero value, so they are non-basic variables.

Step 4: Solve the modified LPP by the simplex method.

While making iterations, using the simplex method, one of the following three cases may arise:

a. If no artificial variable remains in the basis and the optimality condition is satisfied, then the solution is an optimal feasible solution.
b. If at least one artificial variable appears in the basis at zero level and optimality condition is satisfied, then the solution is optimal feasible solution. But it is a degenerate solution. The constraints are consistent though redundancy may exist in them. Redundancy means that the problem has more than the required number of constraints.
c. If at least one artificial variable appears in the basis at a non-zero level (with +ve value in b-column) and the optimality condition is not satisfied, then the original problem has

no feasible solution. Therefore, the original problem doesn't have feasible solution because it contains artificial variable (a very large penalty M).

Remarks:

1. Slack variables are added to the constraints of (≤) type and subtracted from the constraints of (≥) type.
2. Artificial variables are added to the constraints of (≥) and (=) type. Equality constraints require neither slack nor surplus variables.
3. Variables, other than the artificial variables, once driven out in iteration, may re-inter in a subsequent iteration.

Example:

Max $Z = 3x_1 - x_2$
Subject to:
$2x_1 + x_2 \leq 2,$
$x_1 + 3x_2 \geq 3,$
$x_2 \leq 4,$
$x_1, x_2 \geq 0.$

Step 1: Set up the problem in standard form

Max $Z = 3x_1 - x_2 + 0s_1 + 0s_2 + 0s_3 - MA_1,$
Subject to:
$2x_1 + x_2 + s_1 = 2,$
$x_1 + 3x_2 - s_2 + A_1 = 3,$
$x_2 + s_3 = 4,$
$x_1, x_2, s_1, s_2, s_3, A_1 \geq 0.$

Step 2: Find initial basic feasible solution

Substituting $x_1 = x_2 = s_2 = 0$, then the initial b.f.s obtained is: $s_1 = 2$, $A_1 = 3$, $s_3 = 4$, $Z = -3M$

Step 3: perform optimality test
Since the Z_j- Cj is -ve under some variable columns the solution is not optimal and can be improved

Step 4: Iterate in to the optimal solution
Successive iteration yields the following tables

Table 1.

	C_j	3	-1	0	0	0	-M		
C_B	BV	x_1	x_2	s_1	s_2	S_3	A_1	b	r
0	s_1	2	1	1	0	0	0	2	2
-M	A_1	1	(3)	0	-1	0	1	3	1
0	S_3	0	1	0	0	1	0	4	4

	$Z_j=\sum C_B a_{ij}$	-M	-3M	0	M	0	-M	-3M	$\Rightarrow$i.s
	$\Delta j = Z_j - C_j$	-M-3	-3M+1	0	M	0	0		

↑K

Table 2.

	C_j	3	-1	0	0	0		
C_B	BV	x_1	x_2	s_1	s_2	S_3	b	r
0	s_1	(5/3)	0	1	1/3	0	1	3/5
-1	x_2	1/3	1	0	-1/3	0	1	3
0	S_3	-1/3	0	0	1/3	1	3	-9
	$Z_j=\sum C_B a_{ij}$	-1/3	-1	0	1/3	0	-1	$\Rightarrow 2^{nd}$ b.f.s
	$\Delta j = Z_j - C_j$	-10/3	0	0	1/3	0		

↑K

Table 3.

	C_j	3	-1	0	0	0		
C_B	BV	x_1	x_2	s_1	s_2	S_3	b	
3	x_1	1	0	3/5	1/5	0	3/5	
-1	x_2	0	1	-1/5	-2/5	0	4/5	
0	S_3	0	0	1/5	2/5	1	16/5	
	$Z_j=\sum C_B a_{ij}$	3	-1	2	1	0	1	
	$\Delta j = Z_j - C_j$	0	0	2	1	0		

Therefore, optimal solution is given by:
$X_1=3/5$, $x_2=4/5$, $Z_{max}=1$

Exercise: solve the f.f. LPP

Max $Z = x_1+2x_2+3x_3- x_4$

Subject to:

$x_1+2x_2+3x_3=15$

$2x_1+x_2+5x_3=20$

$x_1+2x_2+x_3+ x_4 =10$

$x_1, x_2, x_3, x_4 \geq 0$

Hint: the standard form is:

Max $Z = x_1+2x_2+3x_3- x_4-MA_1-MA_2-MA_3$,

Subject to:

$x_1+2x_2+3x_3+0x_4+A_1+0A_2+0A_3=15$

$2x_1+x_2+5x_3+0x_4+0A_1+A_2+0A_3=20$

$x_1+2x_2+x_3+ x_4+0A_1+0A_2+A_3=10$

$x_1, x_2, x_3, x_4, A_1, A_2, A_3 \geq 0$

2.5. Special cases in the simplex method application

1. Tie Breaking in Simplex Method

While operating the simplex iteration, tie may occur and we may face indifference in choosing the entry or the leaving variable. When tie may occur, the following are important to do:

Tie in the choice of entering variables: the non-basic variable that enters the basis is the one that gives the largest per unit improvement in the objective function.

- Variable having maximum –ve value in a maximization problem and the maximum +ve value in the minimization problem in Z_j- Cj row is the entering variable. A tie in the entering variable exists when more than one variable has same largest (or smallest) value. To break the tie any one of them is selected arbitrarily as the entering variable.

Tie in the choice of leaving variables: This is the case when the ratio is the same.

- This tie can be resolved by choosing one of the variables arbitrarily.

2. No leaving basic variable (unbounded objective function)

- This is the case when all elements of the pivot column are –ve or zero that no variable qualifies to be a leaving basic variable.
- In this case no finite solution can be determined. Because one or more of the variables can be increased indefinitely without violating the feasibility.
- The interpretation is that the objective function is not constrained or the value of the objective function Z can increase indefinitely or the model is poorly constructed.

3. Multiple optimal solutions

- This is the case when the LPP have more than one optimal solution.
- In the simplex table if the Z_j- Cj value is zero for a non-basic variable, it indicates that the existence of more than one optimal solution.

4. Infeasible solution

- If all the constraints are not satisfied simultaneously, the model has no feasible solution.
- In the case of big M method if at least one of the artificial variables has positive value and the solution is optimal, then this is the indication of no feasible solution.

2.6.Minimization case of simplex method

For most part of finding solution for minimization problem using simplex method are handled in the same fashion as maximization problem. The three key exceptions are:

- The coefficients of artificial variables (M) have positive sign in the objective function.
- The selection of pivot column (entering variables) is based on the largest positive number (value) in the index (Δ_j) row.
- The solution is optimal when all values in the index (Δ_j) row are non positive.

The other alternative method of solving minimization problem is by converting it to maximization. i.e., Min Z=Max (-Z)

Example

Min $Z = 12x_1+20x_2$

Subject to:

$6x_1 + 8x_2 \geq 100,$

$7x_1 + 12x_2 \geq 120,$

$x_1, x_2 \geq 0.$

Standard form:

Min $Z = 12x_1+20x_2 +0s_1+0s_2+MA_1+MA_2$

Subject to:

$6x_1 + 8x_2 - s_1+A_1=100,$

$7x_1 + 12x_2 - s_2+A_2=120,$

$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0.$

	Cj	12	20	0	0	M	M		
C_B	BV	x_1	x_2	s_1	s_2	A_1	A_2	b	r
M	A_1	6	8	-1	0	1	0	100	25/2
M	A_2	7	(12)	0	-1	0	1	120	10
	$Z_j=\sum C_B a_{ij}$	13M	20M	-M	-M	M	M	220M	↑I.S
	$\Delta j = Z_j - C_j$	13M-12	20M-20	-M	-M	0	0		

↑K

	Cj	12	20	0	0	M	M		
C_B	BV	x_1	x_2	s_1	s_2	A_1	A_2	b	r
M	A_1	4/3	0	-1	2/3	1	0	20	15
20	x_2	7/12	1	0	-1/12	0	1	10	120/7
	$Z_j=\sum C_B a_{ij}$	-35/3+4/3M	20	-M	-5/3+2/3M	M	M	200+20M	
	$\Delta j = Z_j - C_j$	4/3M-1/3	0	-M	2/3M-5/3	0	0		

↑K

	Cj	12	20	0	0	
C_B	BV	x_1	x_2	s_1	s_2	b
12	x_1	1	0	-3/4	1/2	15
20	x_2	0	1	7/16	-3/4	5/4
	$Z_j=\sum C_B a_{ij}$	12	20	-1/4	-9	205
	$\Delta j = Z_j - C_j$	0	0	-1/4	-9	

CHAPTER THREE: DUALITY THEORY AND SENSITIVITY ANALYSIS

3.1. Duality in linear programming

In the context of LPP, duality implies that each linear programming problem can be analysed using two different ways which have equivalent solutions. The original problem (primal) can be converted into dual by transposing the rows and columns of the algebraic statement of the primal problem.

The various useful aspects of duality:

- If the primal contains large number of rows (constraints) and a smaller number of columns (variables), converting into dual reduces computational procedures.
- Gives additional information as to how the optimal solution changes as a result of changes in coefficients.
- Calculation of the dual checks the accuracy of the primal.
- Duality is used to solve LPP in which the initial solution is infeasible.

The original or primal problem	The dual problem
Max Z= $\sum_{j=1}^{n} C_j X_j$	Min Z= $\sum_{j=1}^{n} b_j y_j$
Subject to	Subject to
$\sum_{j=1}^{n} a_{ij} X_j \leq b_i$	$\sum_{j=1}^{n} a_{ji} y_i \geq c_j$
$X_j \geq 0$	$y_i \geq 0$

Rules of transformation of the primal to the dual

The following rules are important to note when we develop a dual problem:

1. If the primal contains n variables and m constraints, the dual will contain m variables and n constraints.
2. The maximization problem in the primal becomes the minimization problem in the dual and vice versa.
3. The maximization problem has (≤) constraints while the minimization problem has (≥) constraints.
4. Constraints of (≤) type in the primal become (≥) type in the dual and vice versa.
5. The coefficient matrix of the constraints of the dual is the transpose of the primal.
6. The constants c_1, c_2… c_n in the objective function of the primal appear in the constraints of the dual.

7. The constants b_1, b_2… b_n in the constraints of the primal appear in the objective function of the dual.

Example:

Primal	Dual
Max $Z = 2x_1+3x_2$	Min $Z' = 18y_1+19y_2$
Subject to:	Subject to:
$4x_1 + 3x_2 \leq 18$,	$4y_1 + 5y_2 \geq 2$,
$5x_1 +2x_2 \leq 19$,	$3y_1 + 2y_2 \geq 3$,
$x_1, x_2 \geq 0$.	$y_1, y_2 \geq 0$.

Economic Interpretation of Dual Problem

a. The objective function of the dual problem is given by the equation:

$Z' = b_1y_1 +b_2y_2 +b_3y_3$ - - -+ b_my_m

- Each b_iy_i is the current contribution to the objective function by having b_i units of resource i available for primal problem. Thus, y_i interpreted as the contribution to the objective function per unit of resource i when the current set of basic variables is used to obtain the primal solution.
- In other words, y_i's are the shadow prices.

b. The constraint part which can be specified as: $\sum a_{ji}y_i \geq Cj$.

- Since each unit of activity j in the primal problem consumes a_{ij} unit of resource i, then $\sum a_{ji}y_i$ is interpreted as current contribution to the objective function of the mix of resources that would be consumed if one unit of activity j were produced.
- Cj is interpreted as: per unit contribution to the objective function from activity j.

❖ Therefore, $\sum a_{ji}y_i \geq Cj$ can be interpreted as the current contribution to the objective function of resources must be at least as much as the unit contribution to the objective function from activity j, otherwise the use of resources would not be the best possible use.
Yi ≥ 0, meaning that the current contribution to the objective function of resource i must be non-negative otherwise it is better not to use the resource at all. If we are using the resource i while its contribution to the objective function is negative, we are devastating the resource for nothing.

3.1.1. Solving a Dual Linear Programming

The methods of solving the dual linear programming are similar with that of the original linear programming problems. We can use either graphic method or simplex method.

❖ From the optimal simplex table of the dual problem, it is possible to read the optimal solution of the primal problem. That is, the shadow prices of the dual problem indicate the optimal values of original variables.

Example:

Primal

Max $Z = 2x_1+3x_2$

Subject to:

$4x_1 + 3x_2 \leq 18,$

$5x_1 +2x_2 \leq 19,$

$x_1, x_2 \geq 0.$

Dual

Min $Z' = 18y_1+19y_2$ OR Max $(-Z') = -18y_1-19y_2$

Subject to:

$4y_1 + 5y_2 \geq 2,$

$3y_1 + 2y_2 \geq 18,$

$y_1, y_2 \geq 0.$

Standardize the problem

Max $Z = 2x_1+3x_2 +0s_1+0s_2$

Subject to:

$4x_1 + 3x_2 +s_1= 18,$

$5x_1 +2x_2 +s_2= 19,$

$x_1, x_2, s_1, s_2 \geq 0.$

Max $(-Z') = -18y_1-19y_2 +0s_1+0s_2 -MA_1-MA_2$

Subject to:

$4y_1 + 5y_2 -s_1 +A_1 = 2,$

$3y_1 + 2y_2 - s_2 +A_2 =18,$

$y_1, y_2, s_1, s_2, A_1, A_2 \geq 0.$

Solution for the primal

$x_1 =0$, $x_2 =6$

$s_1 =0$, $s2 =7$

Index row

$s_1 =1$, $s2 =0$

$x_1 =2$, $x_2 =0$

Solution for the dual

$y_1 =1$, $y_2 =0$

$s_1 =2$, $s2 =0$

$y_1 =0$, $y_2 =7$

$s_1 =0$, $s_2 =6$

3.2. Sensitivity analysis

One of the assumptions of a LPP is the assumptions of certainty. This assumption implies that the coefficients of a LPP are completely known (determined) and do not change during the period being studied.

- Cj- per unit (profit or cost) contribution of each decision variable
- b_i - availability of resources
- a_{ij} – technical coefficients (per unit resource consumptions or production of each decision variables) are constant and known with certainity. However, in reality, these coefficients are subject to change with time or error.

If such changes will be occurred, there should be a means to check for how long the present optimal solution continues as optimal. The method of evaluating the degree to which the present optimal solution is continued as optimal is called sensitivity analysis.

Sensitivity analysis is concerned with the study of ‘sensitivity’ of the optimal solution of LP problem with changes in parameters. In this case, we are going to determine the range (both lower and upper) over which the linear programming model parameters can change with out affecting the current optimal solution.

i. Changes in C_j of variables

One of the important parameters of linear programming problem is the coefficient of objective function. The test of the sensitivity of the coefficient of objective function involves finding the range of values within which each Cj can lie without changing the current optimal solution.

a. Changes in coefficient C_j of non-basic variable x_j

Among the variables included in the linear programming problem some are non-basic variables. If C_j is the coefficient of non-basic variable $x_{j:}$

1. it doesn't affect any of C_j values of basic variables.
2. it doesn't affect any of Z_j values.
3. it affects (Z_j- C_j) values.

In addition, if optimality is to be maintained, for maximization LPP, coefficients in the index row should be non-negative $(Z_j - C_j) \geq 0$ for all j.

Example:

Max $Z = 2x_1+3x_2$

Subject to:

$4x_1 + 3x_2 \leq 18$,

$5x_1 +2x_2 \leq 19$,

$x_1, x_2 \geq 0$.

	Cj	2	3	0	0		
C_B	BV	x_1	x_2	s_1	s_2	b	r
0	s_1	4	(3)	1	0	18	
0	s_2	5	2	0	1	19	
	$Z_j=\sum C_B a_{ij}$	0	0	0	0	0	
	$\Delta_j = Z_j - Cj$	-2	-3	0	0		

	Cj	2	3	0	0		
C_B	BV	x_1	x_2	s_1	s_2	b	r
3	x_2	4/3	1	1/3	0	6	
0	s_2	7/3	0	-2/3	1	7	
	$Z_j=\sum C_B a_{ij}$	4	3	1	0	18	
	$\Delta_j = Z_j - Cj$	2	0	1	0		

4. In the final table, x_1 and s_1 are non-basic variable whereas x_2 and s_2 are basic variables.
5. C_1=2 is the coefficient of non-basic variable x_1. Let c_1 is subject to change by Δc_1

$C_1' = C_1 + \Delta C_1$

To maintain the optimality condition: $\Delta_j\ (Z_j - C'_j) \geq 0$

$(Z_j - C'_j) \geq 0$

$Z_j - (C_j + \Delta C_1) \geq 0$

$Z_1 - C_1 \geq \Delta C_1$

$4-2 \geq \Delta C_1$

$2 \geq \Delta C_1$
$\Delta c_1 \geq (-\infty, 2]$

Therefore, the range over which the parameter (C_1) can change without affecting the current optimal solution (0, 6) is (-∞, 4].

Simply for non-basic variable x_k, the coefficient (C_j) can change without affecting the optimal solution if $Z_j - C_j \geq \Delta C_k$.

b. Change in Coefficient C_j of Basic Variable X_j

Change in the coefficients of basic variable can affect Z_j, Δ_j and the value of the objective function.

❖ For the above table (x_2 is basic variable). And suppose C_2 is changed by ΔC_2.

i.e., $C_2' = 3 + \Delta C_2$

Cj		2	3	0	0		
C_B	BV	x_1	x_2	s_1	s_2	b	r
$3+\Delta c_2$	x_2	4/3	1	1/3	0	6	
0	s_2	7/3	0	-2/3	1	7	
	$Z_j = \sum C_B a_{ij}$	$4+4/3\Delta c_2$	$3+\Delta c_2$	$1+1/3\Delta c_2$	0	$18+6\Delta c_2$	
	$\Delta_j = Z_j - Cj$	$2+4/3\Delta c_2$	0	$1+1/3\Delta c_2$	0		

The current solution will be optimal if $\Delta_j (Z_j - c'_j) \geq 0$.
i.e., $2+4/3\Delta c_2 \geq 0 \Rightarrow \Delta c_2 \geq -3/2$

$1+1/3\Delta c_2 \geq 0 \Rightarrow \Delta c_2 \geq -3$ *$c_2' = 3 + (-3/2) = 1.5$*

The common value is $\Delta c_2 \geq -3/2 \Rightarrow \Delta c_2 = [-3/2, +\infty)$.

Therefore, the range over which the parameter (c_2) can change without affecting the current optimal solution (0, 6) is [1.5, +∞).

ii. The variation in the right-hand side of constraints (b_i's)

In LPP b_i's represent capacity or availability of resources, which are critical to the selection of optimal alternative solution.

a. Sensitivity of b_i's of fully utilized resources.

If resources are fully utilized, a non-zero shadow price will appear in the Δ_j row with these constants (corresponding to slack or surplus variables). The value indicates that by how much the value of the objective function will change for one additional unit of that resource.

Based on the above optimal table, the first resource is fully utilized. Therefore, its shadow price is 1.

- If we add one additional unit of the first resource, the value of Z will increase by one unit.

Suppose the first resource increase by one unit and become 19 (b_1=19), the new solution would be:

$$\begin{pmatrix} x_2 \\ s_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix} + 1\begin{pmatrix} \frac{1}{3} \\ \frac{-2}{3} \end{pmatrix} = \begin{pmatrix} \frac{19}{3} \\ \frac{19}{3} \end{pmatrix}$$

	Cj	2	3	0	0		
C_B	BV	x_1	x_2	s_1	s_2	b	r
3	x_2	4/3	1	1/3	0	19/3	
0	s_2	7/3	0	-2/3	1	19/3	
	$Z_j=\sum C_B a_{ij}$	4	3	1	0	19	
	$\Delta_j = Z_j - C_j$	2	0	1	0		

So the new solution will be: x_1=0, x_2=19/3 and Z=19

The change in the value of b_i's of fully utilized resources, the current optimal solution will change. Therefore, there is no need to run sensitivity analysis for fully utilized resources.

b. Sensitivity of b_i's not fully utilized resources.

If resources are not fully utilized, a zero shadow price will appear in the Δ_j row with these constants (corresponding to slack or surplus variables). The value indicates that the value of the objective function will not change for any additional unit of that resource.

Let $b_2' = b_2 + \Delta b_2$. In order to get feasible solution b_i must be ≥ 0 ($b_i \geq 0$).

$$\begin{pmatrix} x_2 \\ s_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix} + \Delta b_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \geq 0$$

$b_2' = b_2 + \Delta b_2$

$b_2' = 19 + (-7) = 12$

$$\begin{pmatrix} 6 + \Delta b_2 \\ 7 + \Delta b_2 \end{pmatrix} \geq 0$$

$7 + \Delta b_2 \geq 0$

$\Delta b_2 \geq -7 \Rightarrow \Delta b_2 = [-7, +\infty)$.

Therefore, the range over which the parameter (b_2) can change without affecting the current optimal solution (0, 6) is [12, +∞).

Exercise:

$$\text{Max } Z = 3x_1 + x_2 - x_3$$

Subject to:

$$2x_1 + x_2 + x_3 \leq 8,$$
$$4x_1 + x_2 - x_3 \leq 10,$$
$$x_1, x_2, x_3 \geq 0.$$

a. Find its optimal solution
b. Find the range of values of b_2 for which the current optimal solution remain optimal
c. If $b_2=2$, what is the new optimal solution.